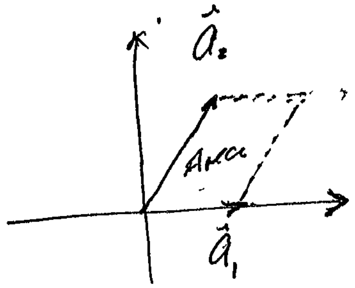
Determinants as Area or Volume

Theorem 9.

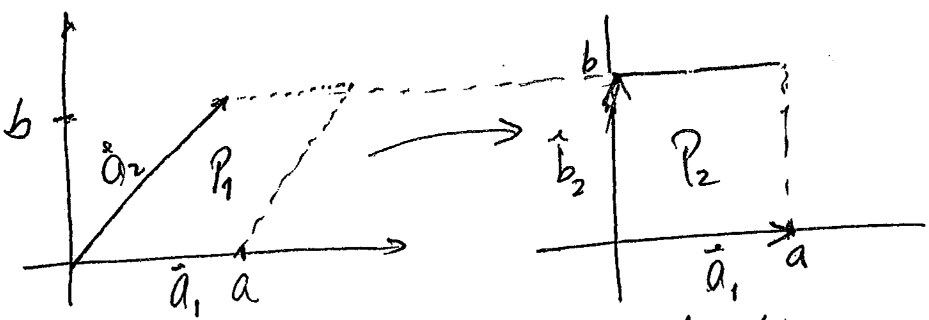
i) $A_{2\times 2}$, $A = [\vec{a}_1, \vec{a}_2] \Rightarrow$ Area of Parallelogram determined by $\vec{a}_1, \vec{a}_2$ $= |\det A|$



ii) $A_{3\times 3}$, $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3] \Rightarrow$ Area of Parallelepiped determined by $\vec{a}_1, \vec{a}_2, \vec{a}_3$

$= |\det A|$.

Proof.– Consider the parallelograms P_1 and P_2.



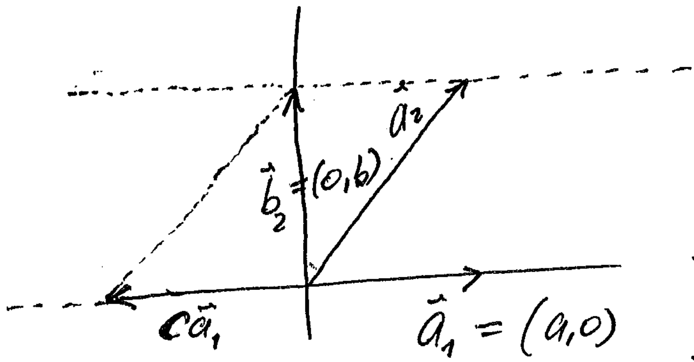
With a common base and the same height

Obviously area $(P_1) =$ area (P_2). $= ab$.

Notice also

if $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = [\vec{a}_1\ \vec{b}_2] \Rightarrow |\det A| = |ab| =$ area $(P_2) =$ area (P_1)

and the vector $\vec{b}_2$ defining P_2 can be obtained by adding a convenient multiple of vector $\vec{a}_1$ to $\vec{a}_2$. In fact,



where $\vec{b}_2 = \vec{a}_2 + c\vec{a}_1$

If $\vec{a}_1 = \begin{pmatrix} a \\ 0 \end{pmatrix}$ and $\vec{a}_2 = \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}$

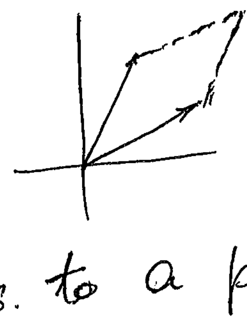
$\Rightarrow \vec{b}_2 = \begin{pmatrix} ca + a_{21} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ a_{22} \end{pmatrix}$ for $c = \frac{-a_{21}}{a}$

,

$$A = \begin{bmatrix} a_{11}=a & a_{21} \\ a_{21}=0 & a_{22} \end{bmatrix} = [\vec{a}_1\ \vec{a}_2] \sim [\vec{a}_1\ \vec{b}_2] = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

and $|\det A| = \det \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = |ab| = \text{area}(P_2) = \text{area}(P_1)$

The matrix $A_{2\times 2}$ whose column vectors define a more general parallelogram



can be reduced to a row equivalent matrix diagonal that corresponds to a parallelogram with defining vectors along the axis, <u>after two row operations.</u>

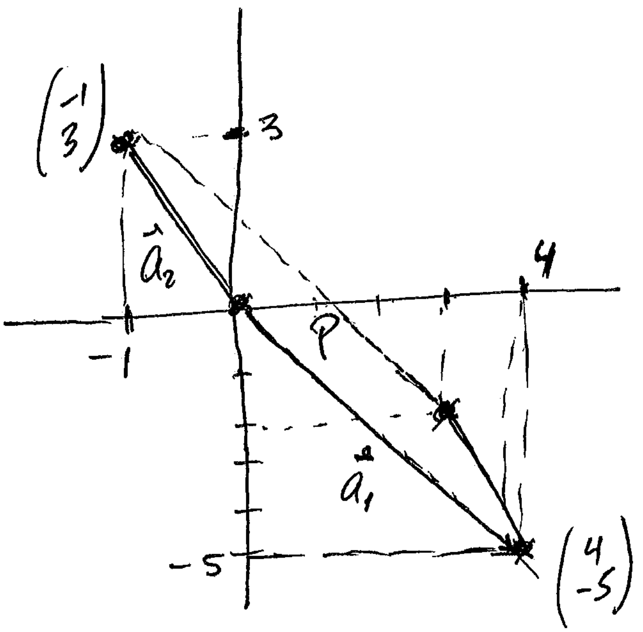
So, in all cases

$$|\det A| = \text{area}(P)$$

P parallelogram determined by columns of A.

It can be proved similarly (considering the three column vectors) for the Parallelepiped.

Exercise # 20   Find the area of the parallelogram whose vertices are: $(0,0)$, $(-1,3)$, $(4,-5)$, and $(3,-2)$.



then

$$A = \begin{bmatrix} 4 & -1 \\ -5 & 3 \end{bmatrix} \Rightarrow \text{Area}(P) = \left| \begin{matrix} 4 & -1 \\ -5 & 3 \end{matrix} \right| = 12 - 5 = 7$$

Ask for Question # 25.

Thm 10.–

i) $T: \mathbb{R}^2 \to \mathbb{R}^2$ lin. transf. determined by $A_{2\times 2}$

If S is a parallelogram in $\mathbb{R}^2$, then

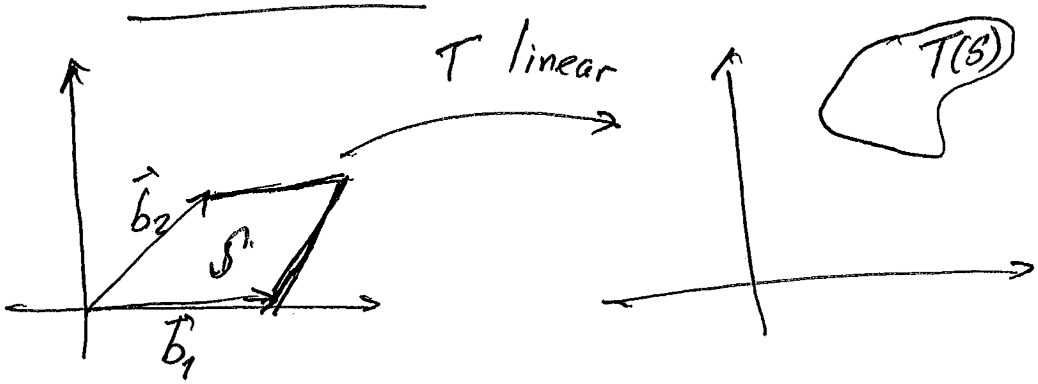
$$\text{area}(T(S)) = |\det A| \cdot \text{area}(S).$$

ii) $T: \mathbb{R}^3 \to \mathbb{R}^3$ L.T. determined by $A_{3\times 3}$

If S is a parallelepiped in $\mathbb{R}^3$, then

$$\text{Vol}(T(S)) = |\det A| \cdot \text{Vol}(S).$$

Proof.– Case in $\mathbb{R}^2$.



$$S = \{ s_1 \hat{b}_1 + s_2 \hat{b}_2 : 0 \le s_1 \le 1, \quad 0 \le s_2 \le 1 \}$$

then for any point in S its image is given by

$$T(s_1\vec{b}_1 + s_2\vec{b}_2) = s_1 T(\vec{b}_1) + s_2 T(\vec{b}_2).$$

Therefore, the parallelogram S is mapped into another parallelogram determined by the vectors $T(\vec{b}_1)$ and $T(\vec{b}_2)$.

$$T(S) = \{ s_1 \underset{A\vec{b}_1}{\underset{\parallel}{T(\vec{b}_1)}} + s_2 \underset{A\vec{b}_2}{\underset{\parallel}{T(\vec{b}_2)}} : 0 \le s_1, s_2 \le 1 \}.$$

and using thm 9.

$$\text{Area of } T(S) = \left| \det [A\vec{b}_1 \; A\vec{b}_2] \right| \underset{\text{def prod. } AB}{=}$$

$$= |\det(AB)| = \det A \det B = \det A \text{ area}(S).$$

Similar for Parallelepipeds in $\mathbb{R}^3$.

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thm 10 can be generalized to arbitrary finite area and volume regions in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

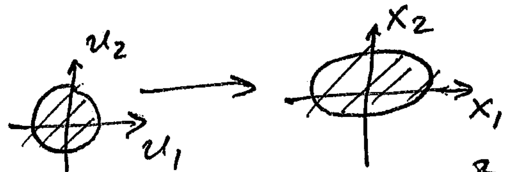
Example 5.

For $a, b > 0$, find the area of the region bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$

The idea is to define a transformation T

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$



$0 \le u_1^2 + u_2^2 \le 1$ $\qquad$ $0 \le \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \le 1$

$T(\vec{u}) = A\vec{u} = \vec{x}$ that transform the unit circle into the ellipse.

If we define $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

then $A\vec{u} = \hat{x} \iff \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

or $\left.\begin{aligned} a u_1 &= x_1 \\ b u_2 &= x_2 \end{aligned}\right\} \iff u_1 = \frac{x_1}{a}, \quad u_2 = \frac{x_2}{b}$

Thus, $0 \le u_1^2 + u_2^2 \le 1 \iff 0 \le \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \le 1$

Therefore,

$$\text{Area (ellipse)} = \text{Area of } T\text{(unit circle)}$$

$$= \det A \cdot \text{Area}\left(\overset{\text{unit}}{\text{circle}}\right) = \underline{\underline{ab \cdot \pi}}.$$

3.3 Cramer's rule

Consider the 2×2 system $A\vec{x}=\vec{b}$, $A_{2\times2}$

or

$$\begin{aligned} \text{Eq.1:} \quad & a_{11}x_1 + a_{12}x_2 = b_1 \\ \text{Eq.2:} \quad & a_{21}x_1 + a_{22}x_2 = b_2 \end{aligned}$$

To solve eliminate x_1 $\begin{pmatrix} -a_{21}\times \text{Eq.1} \\ a_{11}\times \text{Eq.2} \end{pmatrix}$

$$-a_{21}\cancel{a_{11}x_1} - a_{21}a_{12}x_2 = -a_{21}b_1$$

$$\cancel{a_{11}a_{21}x_1} + a_{11}a_{22}x_2 = a_{11}b_2$$

$$(a_{11}a_{22}-a_{21}a_{12})x_2 = a_{11}b_2 - a_{21}b_1 \qquad (*)$$

Therefore, if $a_{11}a_{22}-a_{21}a_{12}\neq 0$

$$\boxed{x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22}-a_{21}a_{12}}} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Similarly,

$$\boxed{x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22}-a_{21}a_{12}}} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Let's introduce a notation here:

$$A_2(\vec{b}) = [\vec{a}_1\ \vec{b}] = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$$

$$I_2(\vec{x}) = [\hat{e}_1\ \vec{x}] = \begin{bmatrix} 1 & x_1 \\ 0 & x_2 \end{bmatrix} \Rightarrow \det I_2(\vec{x}) = x_2$$

Then, equ. (*) can be written as

$$\det A \ \det \overset{= x_2}{I_2(\vec{x})} = \det A_2(\vec{b}).$$

Notice also,

$$A \cdot I_2(\vec{x}) = A_2(\vec{b})$$

Since
$$A \cdot I_2(\vec{x}) = A\ [\hat{e}_1\ \vec{x}] =$$
$$= [A\hat{e}_1\ A\vec{x}] = [\vec{a}_1\ \vec{b}] = A_2(\vec{b})$$

This result for 2×2 matrices can be generalized for $n\times n$ matrices

Therefore, in the new notation

$$x_2 = \frac{\det A_2(\vec{b})}{\det A}, \quad x_1 = \frac{\det A_1(\vec{b})}{\det A}$$

In fact, this is the statement of thm 7.

Thm 7.- $A_{n\times n}$ invertible

For any $b \in \mathbb{R}^n$, the unique soln. $\vec{x}$ of $A\vec{x}=\vec{b}$ has entries given by

$$x_i = \frac{\det A_i(\vec{b})}{\det A}$$

Proof.- Consider the equation $A\vec{x}=\vec{b}$.
We want to solve for $\vec{x}$.

then,

$$A \cdot I_i(\vec{x}) = [\vec{a}_1 \ldots \overset{A}{\vec{a}_i} \ldots \vec{a}_n][\hat{e}_1 \ldots \overset{i}{\vec{x}} \ldots \hat{e}_n] =$$

$$= [A\hat{e}_1 \ldots \overset{i}{\underset{}{A\vec{x}}}{}^{=\vec{b}} \ldots A\hat{e}_n] = [A\hat{e}_1 \ldots \vec{b} \ldots A\hat{e}_n]$$

$$= [\vec{a}_1 \ldots \vec{b} \ldots \vec{a}_n]$$

thus,

$$A I_i(\vec{x}) = A_i(\vec{b})$$

$$I_i(\vec{x}) = \begin{array}{cc} & \text{ith column} \\ & \begin{bmatrix} 1 & 0 & \cdots & x_1 & \cdots & 0 \\ 0 & 1 & \cdots & x_2 & \cdots & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & 0 & \cdots & x_i & \cdots & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & & & x_n & & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \text{ith row} \\ \\ \\ \end{array} \end{array}$$

cofactor expansion

$\Rightarrow \det I_i(x) = x_i$

$$\Rightarrow \det A \underbrace{\det I_i(\vec{x})}_{= x_i} = \det A_i(\vec{b})$$

$$\Rightarrow \det A \; x_i = \det A_i(\vec{b})$$

$$\Rightarrow \quad x_i = \frac{\det A_i(\vec{b})}{\det A}$$

For a matrix $A_{n\times n}$ given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

There is a Submatrix called $\underline{\underline{A_{ij}}}$ obtained by eliminating row i and column j.

This Submatrix is also called a <u>minor of matrix A</u>.

From A_{ij} is possible to define a Cofactor C_{ij} as

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

We saw that determinant of A. can be evaluated from its cofactors expansion.
In fact,

Along row i : $\det A = a_{i1} C_{i1} + a_{i2} C_{i2} \cdots + a_{in} C_{in}$

Along col j : $\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$.

— Discuss convenient evaluation in the presence of zero entries.

We can now form a new matrix

Cofactor values as

$$M = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

We will call M^T the <u>adjugate of A</u> or <u>its classical adjoint.</u>

and we will prove

Thm 8.- If $A_{n\times n}$ is invertible then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A .$$

Example:

$$\begin{cases} 2x + 3y = -5 \\ 5x - y = 13 \end{cases}$$

$$A = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix} \Rightarrow \det A = -2 - 15 = -17.$$

and

$$x = \frac{\begin{vmatrix} -5 & 3 \\ 13 & -1 \end{vmatrix}}{-17} = \frac{5-39}{-17} = 2$$

$$y = \frac{\begin{vmatrix} 2 & -5 \\ 5 & 13 \end{vmatrix}}{-17} = \frac{26+25}{-17} = \frac{51}{-17} = -3$$

Consider

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ \vdots & & & \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix}$$

Since $AA^{-1} = I_n$

Then,

$$A\,[\vec{d}_1\ \vec{d}_2 \cdots \vec{d}_n] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \vdots \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{bmatrix} = [\hat{e}_1\ \hat{e}_2 \cdots \hat{e}_n]$$

In particular,

$$A\vec{d}_j = \hat{e}_j\,, \qquad A^{-1} = \begin{array}{c} \begin{array}{ccccc} & & j\text{th column} & & \end{array} \\ \begin{bmatrix} d_{11} & \cdots & d_{1j} & \cdots & d_{1n} \\ d_{21} & & d_{2j} & & d_{2n} \\ \vdots & & \vdots & & \vdots \\ d_{i1} & & d_{ij} & & \vdots \\ d_{n1} & & d_{nj} & & d_{nn} \end{bmatrix} \end{array}$$

Using Cramer's rule to find the entries of vector $\vec{d}_j$

$$d_{ij} = \frac{\det A_i(\hat{e}_j)}{\det A}$$

Now,

$$A_i(\hat{e}_j) = \begin{array}{c} \text{ith column} \\ \left[\vec{a}_1 \ \vec{a}_2 \cdots \hat{e}_j \cdots \vec{a}_n\right] \end{array}$$

$$= \begin{array}{cc} \begin{array}{ccccc} & & \text{ith col.} & & \end{array} & \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & 0 & \cdots & a_{1n} \\ a_{21} & a_{22} & & 0 & & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{j1} & a_{j2} & & 1 & & a_{jn} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & & 0 & & a_{nn} \end{bmatrix} & \begin{array}{c} \\ \\ \\ j\text{th row} \\ \\ \\ \end{array} \end{array}$$

Therefore,

Using cofactor expansion along column i.

$$\det A_i(\hat{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji}$$

$$A_{ji} = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1,i-1} & a_{1,i+1} & \cdots & a_{1n} \\ \vdots & & & & \\ a_{j-1,1} & \cdots a_{j-1,i-1} & \cdots & & a_{j-1,n} \\ a_{j+1,1} & & & & \\ a_{n1} & \cdots & & & a_{nn} \end{bmatrix}$$

↳ obtained deleting row j and col i.

Then

$$d_{ij} = \frac{C_{ji}}{\det A}$$

and

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & & C_{nn} \end{bmatrix}$$

= adjugate or Classical adjoint. Also called $\operatorname{adj} A$.

<u>Thm 8.</u>- $A_{n\times n}$ invertible. Then

$$\boxed{A^{-1} = \frac{1}{\det A} \operatorname{adj} A}$$

Exercise 3.3

#14 $$A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

We want to find the inverse of A using the formula given by

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & \vdots & & C_{n2} \\ \vdots & & & \vdots \\ C_{1n} & C_{2n} & & C_{nn} \end{bmatrix}$$

where $\boxed{C_{ij} = (-1)^{i+j} A_{ij}}$

We need to compute all the "minors" A_{ij}

So

$A_{11} = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5,$ $\quad A_{21} = \begin{vmatrix} 6 & 7 \\ 3 & 4 \end{vmatrix} = 3$ $\quad A_{31} = \begin{vmatrix} 6 & 7 \\ 2 & 1 \end{vmatrix} = -8$

$A_{12} = \begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} = -2,$ $\quad A_{22} = \begin{vmatrix} 3 & 7 \\ 2 & 4 \end{vmatrix} = -2$ $\quad A_{32} = \begin{vmatrix} 3 & 7 \\ 0 & 1 \end{vmatrix} = 3$

$A_{13} = \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} = -4$ $\quad A_{23} = \begin{vmatrix} 3 & 6 \\ 2 & 3 \end{vmatrix} = -3$ $\quad A_{33} = \begin{vmatrix} 3 & 6 \\ 0 & 2 \end{vmatrix} = 6$

$$\det A \xlongequal{\text{Using cofactor exp. along Column}} 3\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 6 & 7 \\ 2 & 1 \end{vmatrix} = 3(5) + 2(-8) = -1$$

$\therefore$

$$A^{-1} = \frac{1}{(-1)} \begin{bmatrix} A_{11} & -A_{21} & A_{31} \\ -A_{12} & A_{22} & -A_{32} \\ A_{13} & -A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$$